

NON-COMMUTATIVE ODD CHERN NUMBERS AND TOPOLOGICAL PHASES OF DISORDERED CHIRAL SYSTEMS

EMIL PRODAN AND HERMANN SCHULZ-BALDES

ABSTRACT. An odd index theorem for higher odd Chern characters of crossed product algebras is proved. It generalizes the Noether-Gohberg-Krein index theorem. Furthermore, a local formula for the associated cyclic cocycle is provided. When applied to the non-commutative Brillouin zone, this allows to define topological invariants for condensed matter phases from the chiral unitary (or AIII-symmetry) class in the presence of strong disorder and magnetic fields whenever the Fermi level lies in region of Anderson localization.

1. INTRODUCTION

For a classical compact manifold \mathcal{M} of odd dimension d , the odd Chern character pairs with the K_1 group of homotopically equivalent invertible matrices defined over \mathcal{M} , like the even Chern character pairs with the K_0 group of homotopically equivalent idempotents defined over a compact manifold of even dimension [10]. In the odd-dimensional case, the result of the pairing is the odd Chern number:

$$\text{Ch}_d(U) = \frac{(\frac{1}{2}(d-1))!}{d!} \left(\frac{i}{2\pi}\right)^{\frac{d+1}{2}} \int_{\mathcal{M}} \text{Tr} \left((U^{-1} dU)^d \right), \quad (1)$$

which assigns an integer value to the homotopy class $[U]$ of a smooth function U on \mathcal{M} with values in the invertible matrices. This topological invariant is often referred to as generalized winding number because the case $d = 1$ is precisely the winding number. In this case, the Noether-Gohberg-Krein index theorem states that the integer $\text{Ch}_1(U)$ is equal to the index of the Toeplitz operator associated to U . The main mathematical result of this paper is to prove such an index theorem also for the case of odd dimension $d > 1$ and for an adequate generalization of $\text{Ch}_d(U)$ for U lying in a crossed product algebra of covariant operators. This is to be seen as yet another situation where Connes' program of non-commutative geometry [5] can be carried out. More precisely, let $(\Omega, T, \mathbb{Z}^d, \mathbf{P})$ be a C^* -dynamical system given by a compact topological space Ω furnished with an action T of \mathbb{Z}^d and an invariant and ergodic probability measure \mathbf{P} . Associated to it is the reduced crossed product algebra of families of covariant operators. Given a family of covariant (under a projective representation of \mathbb{Z}^d , see Section 3 for details), invertible operators $U = (U_\omega)_{\omega \in \Omega}$ on $\ell^2(\mathbb{Z}^d, \mathbb{C}^N)$ satisfying for some $A, \lambda > 0$ the condition:

$$\int_{\Omega} \mathbf{P}(d\omega) \|\langle \mathbf{x} | U_\omega | \mathbf{y} \rangle\| \leq A e^{-\lambda |\mathbf{x} - \mathbf{y}|}, \quad \mathbf{x}, \mathbf{y} \in \mathbb{Z}^d, \quad (2)$$

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the odd Chern number can be defined by

$$\text{Ch}_d(U) = \frac{i(\pi)^{\frac{d-1}{2}}}{d!!} \sum_{\rho \in S_d} (-1)^\rho \int_{\Omega} P(d\omega) \text{Tr} \langle 0 | \left(\prod_{j=1}^d U_\omega^{-1} i[X_{\rho_j}, U_\omega] \right) | 0 \rangle. \quad (3)$$

where the sum runs over all permutations ρ in the symmetric group S_d and $(-1)^\rho$ denotes their signature, $|0\rangle$ the state at the origin $0 \in \mathbb{Z}^d$ and Tr the trace over the fiber \mathbb{C}^N , and finally X_j , $j = 1, \dots, d$, are the components of the position operator on $\ell^2(\mathbb{Z}^d, \mathbb{C}^N)$ defined by $(X_j \psi)(\mathbf{x}) = x_j \psi(\mathbf{x})$ where $\mathbf{x} = (x_1, \dots, x_d)$. For periodic systems, (3) is the Fourier transform of (1) with \mathcal{M} being the d -dimensional torus. For the definition of the associated Toeplitz operator, let be given an irreducible representation of the complex Clifford algebra \mathcal{C}_d on $\text{Cliff}(d) = \mathbb{C}^{d'}$ with $d' = 2^{\frac{d-1}{2}}$, provided by self-adjoint $\sigma_1, \dots, \sigma_d$ satisfying $[\sigma_i, \sigma_j] = 2\delta_{i,j}$. Then let us introduce the Dirac operator $D = \sum_{j=1}^d X_j \otimes \sigma_j$ acting on the augmented Hilbert space $\mathcal{H} = \ell^2(\mathbb{Z}^d, \mathbb{C}^N) \otimes \text{Cliff}(d)$. Its phase is $F = D/|D|$ and the associated Hardy projection is $E = \frac{1}{2}(F + \mathbf{1})$. The operator $U_\omega \otimes \mathbf{1}$ on \mathcal{H} will also be denoted by U_ω .

Theorem 1.1. *Let d be odd and $U = (U_\omega)_{\omega \in \Omega}$ a covariant family of invertible operators on $\ell^2(\mathbb{Z}^d, \mathbb{C}^N)$ satisfying (2). Then $EU_\omega E$ is almost surely a Fredholm operator on $E\mathcal{H}$ with almost surely constant index given by*

$$\text{Ind}(EU_\omega E) = \text{Ch}_d(U).$$

Our main motivation to prove Theorem 1.1 roots in its application to topological insulators. Solid state physicists are accustomed with the even Chern numbers in dimension $d = 2$ as they play a central role in the theory of the integer quantum Hall effect [4]. More recently, higher even Chern number have entered the theory of topological insulators [7,8,13] and an index theorem similar to Theorem 1.1 has been proved [12]. Also the odd Chern numbers have already appeared in the literature on topological insulators in the chiral unitary class (also called the AIII-symmetry class) [9, 14, 15]. In fact, the ground state of a periodic system in this class can be uniquely characterized by a particular unitary matrix defined over the Brillouin torus (see eq. (7) below) and thus phases of such systems can be labelled by the odd Chern number (1). In [9] the physical space formula (3) was proposed as a phase label for disordered systems in the chiral unitary class. Furthermore, for an explicit 1-dimensional topological model from the chiral unitary class (see Section 2 below), the invariant was evaluated numerically in the presence of strong disorder by implementing techniques from [12] and found to remain quantized and non-fluctuating as the disorder strength was increased, up to a critical disorder strength where the localization length of the system diverges and the invariant sharply changes its value. By Theorem 1.1, these findings are analytically confirmed and extended to higher dimensions covering, in particular, the physically interesting case of disordered systems from the chiral unitary class in dimension $d = 3$. The connection is being made by adapting Bellissard's formalism [3] describing homogeneous solid state systems by suited crossed product algebras. The most important conclusions drawn below are:

- (i) The ground state (Fermi projection) of any short-range homogeneous disordered quantum lattice system described by a Hamiltonian from the chiral unitary class can be uniquely characterized by a canonical family of covariant unitary matrices $U = (U_\omega)_{\omega \in \Omega}$.

- (ii) This family $U = (U_\omega)_{\omega \in \Omega}$ satisfies Eq. (2) whenever the Fermi level lies in a region of Anderson localization and therefore the non-commutative odd Chern numbers $\text{Ch}_d(U)$ allow to distinguish different phases of chiral unitary systems. Here the localization regime is synonymous with the Aizenman-Molchanov bound on the fractional-powers of the Green's function [2] which can indeed be proved to hold in a small and intermediate disorder regime for relevant models [6].
- (iii) During continuous deformations of the Hamiltonian, the odd Chern number remains pinned at a quantized value as long as the Fermi level is located in a region of localization. At a phase transition the localization length at the Fermi level has to diverge.

These findings confirm the conclusions drawn from the numerical studies in Ref. [9] on 1-dimensional disordered models from the chiral unitary class, and extend them to arbitrary odd dimensions. They also parallel the classification of even dimensional topological solid state systems in the unitary class (also called Class A, with no symmetry at all) which is the only other class not requiring the use of a real structure. The prime example of a non-trivial topological phase in the unitary class is a two-dimensional quantum Hall effect [4]. Higher even Chern numbers needed for the classification of systems in the unitary class in even dimension $d \geq 4$ were studied in [12]. There are a number of differences between even and odd dimensional cases though that ought to be stressed. While the even Chern number in even dimension results from a pairing of the K_0 element specified by the Fermi projection with an even Fredholm module and thus leading to an even index theorem, the odd Chern number in odd dimension stems from a pairing of the K_1 group of the unitary U (in bijection with the Fermi projection by the chiral symmetry) with an adequate odd Fredholm module providing an odd index theorem.

From a physical point of view there are major differences between the unitary and the chiral unitary class which were already revealed by the numerics of [9]. A non-trivial topological phase of Class A system in even dimension necessarily possesses extended states at some energies above and below the Fermi level (because the even Chern numbers of the Fermi projection vanish in the low and high energy limit). On the other hand, in chiral unitary systems a non-trivial topological invariant can go along with Anderson localization for all energies. Furthermore, the scenarios of phase transitions are quite distinct. In Class A systems, two extended spectral regions above and below the Fermi level migrate towards each other until they collide and annihilate (as for Landau levels in quantum Hall systems). In the chiral unitary class extended states appear only at the Fermi level when crossing a boundary between topological phases.

The paper is organized as follows. Section 2 presents the chiral unitary class of homogeneous aperiodic systems and shows how Theorem 1.1 can be used for their topological classification. Section 3 first provides some background on some concepts of non-commutative geometry, then introduces the Fredholm module whose quantized calculus leads to the odd Chern character and analyzes its summability properties. Section 4 first provides a local formula for the odd Chern character, which then readily implies Theorem 1.1. Section 5 proves some mathematical statements supporting the claims (i), (ii) and (iii) above.

2. TOPOLOGICAL CLASSIFICATION OF HOMOGENEOUS CHIRAL UNITARY SYSTEMS

A quantum system described by a Hamiltonian H on some Hilbert space is said to have a chiral symmetry if for some involutive unitary S

$$SHS = -H, \quad S^*S = S^2 = \mathbf{1}. \quad (4)$$

The system described by H is then said to be in the chiral unitary class (or AIII class of the Cartan classification). The chiral orthogonal and chiral symplectic classes are defined by a further symmetry the definition of which requires a real structure on the Hilbert space, but this will not be considered here. The unitary S has eigenvalues -1 and 1 , both of which are supposed to be infinitely degenerate. We will always work in the spectral representation of S so that it can be assumed to be of the block form

$$S = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}. \quad (5)$$

This induces a grading $\mathcal{H}_+ \oplus \mathcal{H}_-$ of the Hilbert space with summands which are equal $\mathcal{H}_+ = \mathcal{H}_-$. In this grading, a Hamiltonian with chiral symmetry (4) is of the form

$$H = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}, \quad (6)$$

with A being an operator on \mathcal{H}_\pm . An immediate consequence of the chiral symmetry is that the energy spectrum is invariant under the reflection $E \leftrightarrow -E$. For a variety of physical reasons, the Fermi level is always fixed to be the reflection symmetric point $E = 0$. Hence the Fermi projection is $P = \chi(H < 0)$ where χ denotes the characteristic function. The sign function $Q = \mathbf{1} - 2P$ of the Hamiltonian is often called the flat band version of H . It is an odd function of H and therefore it also satisfies the chiral symmetry $SQS = -Q$. Consequently, it is also of the block form

$$Q = \begin{pmatrix} 0 & U \\ U^* & 0 \end{pmatrix}. \quad (7)$$

It will be assumed that $E = 0$ is not an eigenvalue of H so that $Q^2 = \mathbf{1}$. This then implies that U is a unitary operator on \mathcal{H}_\pm . Clearly U determines the Fermi projection, and visa versa. Hence invariants of P can be defined in terms of U and this will be done in the sequel under adequate hypothesis on H , by appealing to Theorem 1.1.

In this work H is restricted to be a odd-dimensional one-particle Hamiltonian in the tight-binding representation with an even number $2N$ of orbitals per site. Hence the Hilbert space is $\ell^2(\mathbb{Z}^d, \mathbb{C}^{2N}) = \ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^{2N}$ with odd d . It is spanned (in Dirac ket notation) by the states $|\mathbf{x}, \alpha\rangle$ for $\mathbf{x} \in \mathbb{Z}^d$ and $\alpha = 1, \dots, 2N$ and operators acting only on the fiber \mathbb{C}^d will carry a hat throughout. Restricting to tight-binding models is no restriction for all practical purposes because for continuum systems it results from a Wannier function representation, see *e.g.* [11]. The unitary S will be supposed to be local, namely of the form $S = \mathbf{1} \otimes \hat{S}$ for some matrix \hat{S} acting on \mathbb{C}^{2N} . It is necessarily also unitary and involutive so that its spectrum is contained in $\{-1, 1\}$. It will always be assumed that the multiplicities of 1 and -1 as eigenvalues of \hat{S} are equal (to N) so that \hat{S} is of the same block form (5) as S with entries given by the identity matrix of size N . Actually, if these blocks were not of same size, the Hamiltonian would have a flat band at zero energy, as can easily check for a periodic H by means of the Bloch decomposition discussed below.

Furthermore, H and Q are, of course, of the block form (6) and (7) with operators A and U acting on $\mathcal{H}_\pm = \ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^N$. In the following, first follows a discussion of a periodic Hamiltonian H_0 and then of a disordered (covariant) Hamiltonian H_ω .

2.1. Periodic chiral systems. The generic translation invariant lattice Hamiltonian on $\ell^2(\mathbb{Z}^d, \mathbb{C}^{2N})$ is of the form:

$$(H_0\psi)(\mathbf{x}) = \sum_{\mathbf{y} \in \mathbb{Z}^d} \hat{t}_{\mathbf{x}-\mathbf{y}} \psi(\mathbf{y}), \quad \mathbf{x} \in \mathbb{Z}^d, \quad (8)$$

where each $\hat{t}_{\mathbf{a}}$ is an $2N \times 2N$ matrix with complex entries and $\hat{t}_{-\mathbf{a}} = (\hat{t}_{\mathbf{a}})^*$. Throughout a finite hopping-range condition will be assumed to hold, namely that $\hat{t}_{\mathbf{a}}$ is non-zero only for a finite number of $\mathbf{a} \in \mathbb{Z}^d$. The chiral symmetry of H_0 is guaranteed if and only if $\hat{S} \hat{t}_{\mathbf{a}} \hat{S} = -\hat{t}_{\mathbf{a}}$. Due to translational symmetry, the discrete Fourier transform $\mathcal{F} : \ell^2(\mathbb{Z}^d, \mathbb{C}^{2N}) \rightarrow L^2(\mathbb{T}^d, \mathbb{C}^{2N})$ leads to a direct integral representation of H_0 , namely $\mathcal{F} H_0 \mathcal{F}^* = \int^\oplus d\mathbf{k} H_0(\mathbf{k})$ with Bloch Hamiltonians given by $2N \times 2N$ matrices:

$$H_0(\mathbf{k}) : \mathbb{C}^{2N} \rightarrow \mathbb{C}^{2N}, \quad H_0(\mathbf{k}) = \sum_{\mathbf{a} \in \mathbb{Z}^d} \hat{t}_{\mathbf{a}} e^{i\mathbf{a} \cdot \mathbf{k}}.$$

If the system is an insulator, zero energy 0 lies in a gap of H_0 . Then the Fermi projection P_0 and flat band version Q_0 have similar direct integral representation. As $S = \mathbf{1} \otimes \hat{S}$ is local, both $H_0(\mathbf{k})$ and $Q_0(\mathbf{k})$ are off-diagonal block matrices, similar as in (6) and (7) respectively. The off diagonal blocks of $Q_0(\mathbf{k})$ are $N \times N$ unitary matrices $U_0(\mathbf{k})$ depending analytically on \mathbf{k} . For these unitaries the classical odd Chern number can be defined by (1) with $\mathcal{M} = \mathbb{T}^d$. That this provides an integer that allows to distinguish periodic systems in the chiral unitary class was proposed in the pioneering works [14, 15] where it was referred to as the generalized winding number. Hence the classification of periodic chiral systems can be solely based on the odd Chern character (1) of classical differential topology. As will be seen shortly, this is not sufficient for aperiodic systems. Before going on, let us provide a simple periodic model in dimension $d = 1$ having a non-trivial odd Chern number and thus a topological phase.

Example 2.1. In [9] the Hamiltonian

$$(H_0\psi)(x) = \frac{1}{2}(\hat{\sigma}_1 + i\hat{\sigma}_2)\psi(x+1) + \frac{1}{2}(\hat{\sigma}_1 - i\hat{\sigma}_2)\psi(x-1) + m\hat{\sigma}_2\psi(x), \quad (9)$$

acting on $\ell^2(\mathbb{Z}, \mathbb{C}^2)$ was considered. Here $\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3$ are the Pauli matrices. It has the chiral symmetry (4) with $S = \mathbf{1} \otimes \hat{\sigma}_3$. The odd Chern number $\text{Ch}_1(U_0)$ reduces to the classical winding number and can be computed explicitly to be $\text{Ch}_1(U_0) = 1$ for $m \in (-1, 1)$, and $\text{Ch}_1(U_0) = 0$ otherwise. The spectral gap of the model closes exactly at $m = \pm 1$ where the invariant switches between quantized values. \diamond

2.2. Homogeneous chiral systems. The periodicity of a one-particle model may be broken by a magnetic field and a random aperiodic perturbation. Typically one then has not only one tight-binding Hamiltonian on $\ell^2(\mathbb{Z}^d, \mathbb{C}^{2N})$, but rather a family of $(H_\omega)_{\omega \in \Omega}$ indexed by a parameter from a compact probability space (Ω, \mathbf{P}) of disorder configurations. These configurations can be shifted in physical space so that there is a group action T of \mathbb{Z}^d on Ω by homeomorphisms. The probability measure \mathbf{P} is supposed to be invariant and ergodic w.r.t. T , so that indeed all the

ingredients of a C^* -dynamical system are given. The Hamiltonians are then of the form

$$(H_\omega \psi)(\mathbf{x}) = \sum_{\mathbf{y} \in \mathbb{Z}^d} e^{i\mathbf{y} \wedge \mathbf{x}} \hat{t}_{\mathbf{x}, \mathbf{y}}(\omega) \psi(\mathbf{y}), \quad (10)$$

where \wedge is an anti-symmetric bilinear form incorporating the effect of a constant magnetic field by means of a Peierls phase-factor and the hopping matrices $\hat{t}_{\mathbf{x}, \mathbf{y}}(\omega)$ depend continuously on ω , satisfy

$$\hat{t}_{\mathbf{x}-\mathbf{a}, \mathbf{y}-\mathbf{a}}(\omega) = \hat{t}_{\mathbf{x}, \mathbf{y}}(T_{\mathbf{a}}\omega),$$

and are, moreover, of finite range, namely vanish uniformly for $|\mathbf{x} - \mathbf{y}| > R$ for some $R > 0$. This implies that the collection $(H_\omega)_{\omega \in \Omega}$ of Hamiltonians defines a covariant family of operators, in the sense that

$$V_{\mathbf{a}} H_\omega V_{\mathbf{a}}^{-1} = H_{T_{\mathbf{a}}\omega}, \quad (11)$$

for the magnetic translations defined by

$$V_{\mathbf{a}} \psi(\mathbf{x}) = e^{i\mathbf{a} \wedge \mathbf{x}} \psi(\mathbf{x} - \mathbf{a}). \quad (12)$$

By functional calculus, any function $g(H_\omega)$ is also a covariant operator. In particular, the Fermi projection $P_\omega = \chi(H_\omega < 0)$ is covariant.

In the present context, each Hamiltonian H_ω is supposed to have the chiral symmetry (4) w.r.t. $S = \mathbf{1} \otimes \hat{S}$ described above. By functional calculus, it follows that also any odd function of H_ω has the chiral symmetry and thus, in particular, the flat band version $\mathbf{1} - 2P_\omega$. As in (7) it hence admits again a representation as (7)

$$\mathbf{1} - 2P_\omega = \begin{pmatrix} 0 & U_\omega \\ U_\omega^* & 0 \end{pmatrix}, \quad (13)$$

with a unitary U_ω which satisfies again the covariance relation

$$V_{\mathbf{a}} U_\omega V_{\mathbf{a}}^{-1} = U_{T_{\mathbf{a}}\omega}, \quad (14)$$

where $V_{\mathbf{a}}$ is given by the same formula as above, albeit on the Hilbert space $\ell^2(\mathbb{Z}^d, \mathbb{C}^N)$ with half-dimensional fiber. The family $(U_\omega)_{\omega \in \Omega}$ is thus precisely of the form required by Theorem 1.1. The condition (2) holds either if the Fermi energy $E = 0$ lies in a (almost sure) gap of the spectrum of H_ω , or at least in a region of Anderson localization. This will be proved in Section 5. In conclusion, if the localization condition holds, Theorem 1.1 applies and allows to associate the odd Chern number and associated index to $(U_\omega)_{\omega \in \Omega}$ and thus to the Fermi projection. This provides the phase label discussed in the introduction and its properties are further analyzed in Section 5.

Example 2.2. An easy way to obtain an aperiodic chiral unitary system is to start from the periodic Hamiltonian H_0 given in (8), and then define H_ω via (10) by setting $\hat{t}_{\mathbf{x}, \mathbf{y}}(\omega) = (1 + \lambda \omega_{\mathbf{x}, \mathbf{y}}) \hat{t}_{\mathbf{x} - \mathbf{y}}$ where $\lambda > 0$ is some coupling constant and $\omega_{\mathbf{x}, \mathbf{y}}$ are complex numbers drawn independently according to some probability distribution with compact support (say the unit cube \mathcal{C}) subjected to the constraint $\omega_{\mathbf{x}, \mathbf{y}} = \overline{\omega_{\mathbf{y}, \mathbf{x}}}$. Then $\Omega = \mathcal{C}^{R\mathbb{Z}^{2d}}$ is a compact Tychonov space obtained by taking an infinite product of the unit cube. On it one has the product measure which is then invariant and ergodic w.r.t. to the natural shift action of \mathbb{Z}^d . To present

something concrete, let us write out the disordered one-dimensional model analyzed numerically in Ref. [9]:

$$(H_\omega \psi)(x) = \frac{1}{2}(1 + \lambda \omega_x)[(\hat{\sigma}_1 + i\hat{\sigma}_2)\psi(x+1) + (\hat{\sigma}_1 - i\hat{\sigma}_2)\psi(x-1)] \\ + (m + \lambda' \omega'_x)\hat{\sigma}_2 \psi(x), \quad (15)$$

with ω_x and ω'_x independent random variables uniformly distributed $[-\frac{1}{2}, \frac{1}{2}]$. \diamond

3. ODD FREDHOLM MODULES FOR THE ALGEBRA OF COVARIANT OBSERVABLES

This section briefly reviews the construction of a reduced crossed product algebra and then introduces the Fredholm modules used for the index calculation of the odd Chern character and analyzes summability issues connected with it. Elements of the crossed products are precisely covariant families of operators and therefore such families constructed from the covariant tight-binding Hamiltonians presented in the previous section belong to the crossed product algebra. For this reason and further analogies, Bellissard suggested [3] to call these crossed product algebra in the context of solid state physics also the Non-Commutative Brillouin Zone (NCBZ). Let us start from a C^* -dynamical system $(\Omega, T, \mathbb{Z}^d, \mathbf{P})$ as described in the introduction. The algebra \mathcal{A}_0 is defined as the set $C_c(\Omega \times \mathbb{Z}^d, M_{N \times N})$ of compactly supported, matrix-valued continuous functions on $\Omega \times \mathbb{Z}^d$ furnished with the following algebraic operations:

$$(f + \lambda g)(\omega, \mathbf{x}) = f(\omega, \mathbf{x}) + \lambda g(\omega, \mathbf{x}), \\ (fg)(\omega, \mathbf{x}) = \sum_{\mathbf{y} \in \mathbb{Z}^d} e^{i\mathbf{x} \wedge \mathbf{y}} f(\omega, \mathbf{y}) g(T_{\mathbf{y}}^{-1}\omega, \mathbf{x} - \mathbf{y}), \\ f^*(\omega, \mathbf{x}) = f(T_{\mathbf{x}}^{-1}\omega, -\mathbf{x})^*.$$

Covariant representations π_ω on $\ell^2(\mathbb{Z}^d, \mathbb{C}^N)$ are defined by the fiber-wise actions:

$$\mathcal{A}_0 \ni f \mapsto (\pi_\omega(f))\psi(\mathbf{x}) = \sum_{\mathbf{y} \in \mathbb{Z}^d} e^{i\mathbf{x} \wedge \mathbf{y}} f(T_{\mathbf{y}}^{-1}\omega, \mathbf{y} - \mathbf{x})\psi(\mathbf{y}),$$

where \wedge is an anti-symmetric bilinear on \mathbb{Z}^d . The family $F = (F_\omega)_{\omega \in \Omega}$ of such representations $F_\omega = \pi_\omega$ satisfies the covariance relation $V_a F_\omega V_a^* = F_{T_a \omega}$ with V_a as defined in (12). Inversely, given such a covariant family $F = (F_\omega)_{\omega \in \Omega}$ of finite range operators, there is an associated $f \in \mathcal{A}_0$ defined by $f(\omega, \mathbf{x}) = \langle \mathbf{0} | F_\omega | \mathbf{x} \rangle \in M_{N \times N}$. In particular, the family of covariant Hamiltonians $H = (H_\omega)_{\omega \in \Omega}$ defined in 10 can be identified with an element $h \in \mathcal{A}_0$ via $h(\omega, \mathbf{x}) = \hat{t}_{\mathbf{0}, \mathbf{x}}(\omega)$ (more precisely, the size N is doubled to $2N$, but this difference will be suppressed). On the other hand, the covariant family of unitaries $U = (U_\omega)_{\omega \in \Omega}$ associated to H is in general not of finite range and therefore not in \mathcal{A}_0 , but only an adequate closure of it. The first such closure \mathcal{A} is the completion of \mathcal{A}_0 w.r.t. the C^* -norm:

$$\|f\| = \sup_{\omega \in \Omega} \|\pi_\omega f\|.$$

This is a C^* -algebra also called the reduced twisted crossed product algebra. As \mathcal{A} is stable under the continuous functional calculus, the unitary U lies in \mathcal{A} provided the Fermi level lies in a gap. If there is no gap, U only lies in the von Neumann closure $L^\infty(\mathcal{A}, \mathbf{P})$ of \mathcal{A}_0 under the norm

$$\|f\|_\infty = \mathbf{P}\text{-esssup}_{\omega \in \Omega} \|\pi_\omega(f)\|.$$

This algebra $L^\infty(\mathcal{A}, \mathbf{P})$ is stable under Borel functional calculus and therefore contains U . The algebra \mathcal{A}_0 and its completions become non-commutative manifolds when equipped with non-commutative differential calculus tools, namely integration

$$\mathcal{T}(f) = \int_{\Omega} \mathbf{P}(d\omega) \operatorname{Tr}(f(\omega, \mathbf{0})) ,$$

and derivations

$$(\partial_j f)(\omega, \mathbf{x}) = \imath x_j f(\omega, \mathbf{x}) , \quad j = 1, \dots, d ,$$

where $\mathbf{x} = (x_1, \dots, x_d)$. Associated to the state \mathcal{T} , there is the GNS or Hilbert-Schmidt norm

$$\|f\|_2 = \mathcal{T}(|f|^2)^{\frac{1}{2}} = \left(\int_{\Omega} \mathbf{P}(d\omega) \operatorname{Tr}(f^* f(\omega, \mathbf{0})) \right)^{\frac{1}{2}} .$$

The closure of \mathcal{A}_0 w.r.t. $\|\cdot\|_2$ is the GNS Hilbert space which is, however, not used here. Rather $\|\cdot\|_2$ will be used as norm on $L^\infty(\mathcal{A}, \mathbf{P})$ and subsets of it. Particularly useful in the sequel will be the set \mathcal{A}_{loc} of $f \in L^\infty(\mathcal{A}, \mathbf{P})$ satisfying

$$\int_{\Omega} \mathbf{P}(d\omega) \|f(\omega, \mathbf{x})\| \leq A e^{-\lambda|\mathbf{x}|} , \quad \text{for some } A, \lambda > 0 . \quad (16)$$

We will refer to \mathcal{A}_{loc} as the set of localized observables, due to its link to Anderson localization explained in Section 5. As it is a somewhat non-standard object, so let us collect a few basic properties of \mathcal{A}_{loc} .

Proposition 3.1. *\mathcal{A}_{loc} is a dense $*$ -sub-algebra of $L^\infty(\mathcal{A}, \mathbf{P})$. The expressions*

$$\mathcal{T}(\partial^{\alpha_1} f_1 \dots \partial^{\alpha_k} f_k)$$

are always finite for $f_1, \dots, f_k \in \mathcal{A}_{\text{loc}}$ and multi-indices $\alpha_1, \dots, \alpha_k \in \mathbb{N}_0^d$ defining mixed higher derivatives. Furthermore, given $g_1, \dots, g_k, h_1, \dots, h_k \in L^\infty(\mathcal{A}, \mathbf{P})$, the functional

$$f \in \mathcal{A}_{\text{loc}} \mapsto \mathcal{T} \left(\prod_{j=1}^k f g_j (\partial^{\alpha_j} f) h_j \right)$$

is continuous w.r.t. $\|\cdot\|_2$.

Proof. All facts follow from successive applications of Hölder's inequality and from the observation that $|f(\omega, \mathbf{x})| \leq \|f\|_\infty$, almost surely. \square

Let us recall [5] that an odd Fredholm module (\mathcal{H}, F, π) over a $*$ -algebra \mathcal{A} is by definition a representation π of \mathcal{A} on a Hilbert space \mathcal{H} together with an operator F on \mathcal{H} with the properties 1) $F^* = F$, 2) $F^2 = \mathbf{1}$, and 3) $[F, \pi(a)]$ compact for all $a \in \mathcal{A}$. A Fredholm module is said to be p -summable if the commutators $[F, \pi(a)]$ belong to the p -th Schatten class. The first task is to construct a natural family of $(d+1)$ -summable Fredholm modules of the crossed product algebra \mathcal{A}_0 of compactly supported observables which extends by continuity to the algebra of localized observables. As in the introduction, let $\sigma_1, \dots, \sigma_d$ be the generators of an irreducible representation of the odd complex Clifford algebra \mathcal{C}_d on representation space $\operatorname{Cliff}(d)$ of dimension $d' = 2^{\frac{d-1}{2}}$, namely $\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{i,j}$ and $\sigma_j^* = \sigma_j$

for $i, j = 1, \dots, d$. On the augmented Hilbert space $\mathcal{H} = \ell^2(\mathbb{Z}^d, \mathbb{C}^N) \otimes \text{Cliff}(d)$ the shifted Dirac operator is introduced by

$$D_{\mathbf{x}_0} = \sum_{j=1}^d (X_j + x_{0,j}) \otimes \sigma_j ,$$

where $\mathbf{x}_0 = (x_{0,1}, \dots, x_{0,d}) \in \mathbb{R}^d$. If $\mathbf{x}_0 \in \mathbb{Z}^d$, the Dirac operator has zero modes localized at $\mathbf{x} = -\mathbf{x}_0$ which are eliminated by setting, for example, $(D_{\mathbf{x}_0}\psi)(-\mathbf{x}_0) = \psi(-\mathbf{x}_0)$. This allows to introduce the Dirac phase by

$$F_{\mathbf{x}_0} = \frac{D_{\mathbf{x}_0}}{|D_{\mathbf{x}_0}|} .$$

Proposition 3.2. *The triples $(\mathcal{H}, F_{\mathbf{x}_0}, \pi_\omega \otimes \text{id})$, with \mathbf{x}_0 in the unit cube $\mathcal{C}^d = [0, 1]^d$ and $\omega \in \Omega$, define a family of $(d+1)$ -summable odd Fredholm modules over \mathcal{A}_0 . These Fredholm modules extend almost surely to \mathcal{A}_{loc} and for $f \in \mathcal{A}_{\text{loc}}$*

$$\int_{\Omega} \mathbf{P}(d\omega) \int_{\mathcal{C}^d} d\mathbf{x}_0 \text{Tr} (|[F_{\mathbf{x}_0}, \pi_\omega(f)]|^{d+1}) < \infty . \quad (17)$$

Remark. As it will become apparent from the proof, $d+1$ is the smallest integer for which summability holds. \diamond

Proof. The algebraic properties $F_{\mathbf{x}_0}^* = F_{\mathbf{x}_0}$ and $F_{\mathbf{x}_0}^2 = \mathbf{1}$ are readily checked, so let us focus on the summability and denote the l.h.s. of (17) by I , first for $f \in \mathcal{A}_0$ which, of course, also satisfies (16). Let $|\mathbf{x}, n, j\rangle$ denote an orthonormal basis in \mathcal{H} of vectors localized at $\mathbf{x} \in \mathbb{Z}^d$ on the fiber $n = 1, \dots, N$ and $j = 1, \dots, d'$. Then the projections on site \mathbf{x} are

$$\chi_{\mathbf{x}} = \sum_{n=1}^N \sum_{j=1}^{d'} |\mathbf{x}, n, j\rangle \langle \mathbf{x}, n, j| . \quad (18)$$

With these notations,

$$\begin{aligned} I &= \int_{\Omega} \mathbf{P}(d\omega) \int_{\mathcal{C}^d} d\mathbf{x}_0 \sum_{\mathbf{x} \in \mathbb{Z}^d} \text{Tr} (\chi_{-\mathbf{x}} |[F_{\mathbf{x}_0}, \pi_\omega(f)]|^{d+1}) \\ &= \int_{\mathcal{C}^d} d\mathbf{x}_0 \sum_{\mathbf{x} \in \mathbb{Z}^d} \int_{\Omega} \mathbf{P}(d\omega) \text{Tr} (\chi_0 |[F_{\mathbf{x}_0+\mathbf{x}}, \pi_{T_{\mathbf{x}}\omega}(f)]|^{d+1}) , \end{aligned}$$

where in the second equality the sums and integrals could be exchanged by Tonelli's theorem and the covariance relation was used. Now the T -invariance of \mathbf{P} will be invoked to replace $T_{\mathbf{x}}\omega$ by ω . Then the integral over \mathbf{x}_0 and sum over \mathbf{x} can be combined:

$$I = \int_{\mathbb{R}^d} d\mathbf{x} \int_{\Omega} \mathbf{P}(d\omega) \text{Tr} (\chi_0 |[F_{\mathbf{x}}, \pi_\omega(f)]|^{d+1}) .$$

Next let us expand the operator power:

$$I = \int_{\mathbb{R}^d} d\mathbf{x} \int_{\Omega} \mathbf{P}(d\omega) \sum_{\mathbf{x}_j \in \mathbb{Z}^d} \text{Tr} \left(\prod_{i=0}^d \chi_{\mathbf{x}_i} [F_{\mathbf{x}}, \pi_\omega(f^\#)] \chi_{\mathbf{x}_{i+1}} \right) ,$$

where \mathbf{x}_0 and \mathbf{x}_{2n+2} are fixed at the origin and $\#$ indicates the presence (for i even) or absence (for i odd) of the $*$ -operation. Further note that the sums over

\mathbf{x}_j , $j = 1, \dots, 2n-1$, have a finite number of terms. With the notation $\widehat{\mathbf{v}} = \mathbf{v}/|\mathbf{v}|$, one obtains upon writing out the definition of the Dirac phase

$$I = i^{d+1} \sum_{\mathbf{x}_j \in \mathbb{Z}^d} \int_{\mathbb{R}^d} d\mathbf{x} \operatorname{tr}_\sigma \left(\prod_{i=0}^d (\widehat{\mathbf{x}_i + \mathbf{x} - \mathbf{x}_{i+1} + \mathbf{x}}) \cdot \boldsymbol{\sigma} \right) \times \int_{\Omega} \mathbf{P}(d\omega) \operatorname{tr}_N \left(\prod_{i=0}^d \chi_{\mathbf{x}_i} \pi_\omega(f^\#) \chi_{\mathbf{x}_{i+1}} \right). \quad (19)$$

Here tr_σ and tr_N are the trace over $\operatorname{Cliff}(d)$ and \mathbb{C}^N respectively (with \mathbb{C}^N being the fiber over the origin). Thus a bound on I will follow from bounds on

$$I(\mathbf{x}_1, \dots, \mathbf{x}_d) = \int_{\mathbb{R}^d} d\mathbf{x} \operatorname{tr}_\sigma \left(\prod_{i=0}^d (\widehat{\mathbf{x}_i + \mathbf{x} - \mathbf{x}_{i+1} + \mathbf{x}}) \cdot \boldsymbol{\sigma} \right), \quad (20)$$

and

$$I'(\mathbf{x}_1, \dots, \mathbf{x}_d) = \int_{\Omega} \mathbf{P}(d\omega) \operatorname{tr}_N \left(\prod_{i=0}^d \chi_{\mathbf{x}_i} \pi_\omega(f^\#) \chi_{\mathbf{x}_{i+1}} \right).$$

Let us begin with the latter. Using the definition of π_ω , it takes the form:

$$I'(\mathbf{x}_1, \dots, \mathbf{x}_d) = \int_{\Omega} \mathbf{P}(d\omega) \operatorname{tr}_N \left(\prod_{i=0}^d f^\#(T_{\mathbf{x}_i} \omega, \mathbf{x}_{i+1} - \mathbf{x}_i) \right).$$

By Hölder's inequality

$$|I'(\mathbf{x}_1, \dots, \mathbf{x}_d)| \leq N \prod_{i=0}^d \left(\int_{\Omega} \mathbf{P}(d\omega) \|f^\#(T_{\mathbf{x}_i} \omega, \mathbf{x}_{i+1} - \mathbf{x}_i)\|^{d+1} \right)^{\frac{1}{d+1}}.$$

By change of variable all $T_{\mathbf{x}_i} \omega$ can be changed to ω and using $\|f^\#(\omega, \mathbf{x})\| \leq \|f\|_\infty$ shows

$$|I'(\mathbf{x}_1, \dots, \mathbf{x}_d)| \leq N \|f\|_\infty^d \prod_{i=0}^d \left(\int_{\Omega} \mathbf{P}(d\omega) \|f^\#(\omega, \mathbf{x}_{i+1} - \mathbf{x}_i)\| \right)^{\frac{1}{d+1}}.$$

Now by (16)

$$|I'(\mathbf{x}_1, \dots, \mathbf{x}_d)| \leq N A \|f\|_\infty^d e^{-\xi' \sum_{i=0}^d |\mathbf{x}_{i+1} - \mathbf{x}_i|} \leq N A \|f\|_\infty^d e^{-\xi'' \sum_{i=1}^d |\mathbf{x}_i|},$$

with $\xi' = \xi/(d+1)$ and $\xi'' = \xi'/d$, where in the second inequality the bound $\sum_{i=0}^d |\mathbf{x}_{i+1} - \mathbf{x}_i| \geq \frac{1}{d} \sum_{i=1}^d |\mathbf{x}_i|$ was used.

Next let us turn to $I(\mathbf{x}_1, \dots, \mathbf{x}_d)$ and first of all verify that it is finite. Indeed, in the asymptotic limit of $|\mathbf{x}| \rightarrow \infty$,

$$\widehat{\mathbf{x}_i + \mathbf{x} - \mathbf{x}_{i+1} + \mathbf{x}} \sim |\mathbf{x}|^{-1} [\mathbf{x}_i - \mathbf{x}_{i+1} + (\hat{\mathbf{x}} \cdot (\mathbf{x}_i - \mathbf{x}_{i+1})) \hat{\mathbf{x}}].$$

Hence the integrand in (20) decays as $|\mathbf{x}|^{-(d+1)}$ at infinity, just enough for the integral to converge. Furthermore the integral is continuous in each \mathbf{x}_i . Another key observation is the following scaling property resulting from a change of variables:

$$I(\mathbf{x}_1, \dots, \mathbf{x}_d) = \gamma^d I(\gamma^{-1} \mathbf{x}_1, \dots, \gamma^{-1} \mathbf{x}_d),$$

Using $\gamma = \sum_{i=1}^d |\mathbf{x}_i|$ one then has $\mathbf{x}'_j = \gamma^{-1} \mathbf{x}_j$ lying in the unit ball. Thus

$$|I(\mathbf{x}_1, \dots, \mathbf{x}_d)| \leq \left(\sup_{|\mathbf{x}'_j| \leq 1} I(\mathbf{x}'_1, \dots, \mathbf{x}'_d) \right) \left(\sum_{i=1}^d |\mathbf{x}_i| \right)^d.$$

By compactness the supremum is bounded. Now replacing this and the bound on $I'(\mathbf{x}_1, \dots, \mathbf{x}_d)$ into (19) and carrying out the sum over the \mathbf{x}_j 's shows that I is bounded. As all the above bounds only invoked the norm $\|\cdot\|_\infty$ and the bound (16) it follows that (17) holds for all $f \in \mathcal{A}_{\text{loc}}$. \square

4. CHERN CHARACTER AND ITS PARING WITH THE K_1 GROUP

Associated to each of the odd Fredholm modules $(\mathcal{H}, F_{\mathbf{x}_0}, \pi_\omega)$ over \mathcal{A}_{loc} constructed in the last section, there is in a standard manner [5] associated a quantized calculus, namely a graded algebra (Ω, d) where $\Omega = \bigoplus_{k=0}^d \Omega^k$ with

$$\Omega^k = \text{span} \{ \pi_\omega(f_0)[F_{\mathbf{x}_0}, \pi_\omega(f_1)] \cdots [F_{\mathbf{x}_0}, \pi_\omega(f_k)] \mid f_1, \dots, f_k \in \mathcal{A}_{\text{loc}} \},$$

together with a differential

$$\eta \in \Omega^k \mapsto d\eta = F_{\mathbf{x}_0} \eta - (-1)^k \eta F_{\mathbf{x}_0} \in \Omega^{k+1},$$

and a closed graded trace (also called supertrace)

$$\eta \in \Omega^d \mapsto \text{Tr}'(\eta) = \frac{1}{2} \text{Tr}(F_{\mathbf{x}_0} d\eta).$$

Over the algebra \mathcal{A}_0 , one can associate Connes' cyclic cocycle to each individual Fredholm module from the family $(\mathcal{H}, F_{\mathbf{x}_0}, \pi_\omega)$, namely the $(d+1)$ -linear functional:

$$f_0, \dots, f_d \in \mathcal{A}_0 \mapsto \lambda_d \text{Tr}'(\pi_\omega(f_0)[F_{\mathbf{x}_0}, \pi_\omega(f_1)] \cdots [F_{\mathbf{x}_0}, \pi_\omega(f_d)]) , \quad (21)$$

where $\lambda_d = 2^{-d} i^{d+1}$ is chosen such that (24) holds. Proposition 3.2 implies that for fixed $f_0, \dots, f_d \in \mathcal{A}_{\text{loc}}$ the r.h.s. is almost surely finite. This does *not* imply, though, that almost surely for ω and \mathbf{x}_0 , the $(d+1)$ -linear functional can be extended to all of \mathcal{A}_{loc} . The key is then to define a $(d+1)$ -cyclic cocycle on \mathcal{A}_{loc} by averaging over the family of Fredholm modules:

Proposition 4.1. *The $(d+1)$ -linear functional*

$$\tau_d(f_0, \dots, f_d) = \lambda_d \int_{\mathcal{C}^d} d\mathbf{x}_0 \int_{\Omega} \mathbf{P}(d\omega) \text{Tr}'(\pi_\omega(f_0)[F_{\mathbf{x}_0}, \pi_\omega(f_1)] \cdots [F_{\mathbf{x}_0}, \pi_\omega(f_d)]) \quad (22)$$

is well-defined on \mathcal{A}_{loc} , continuous w.r.t. $\|\cdot\|_2$ and is a cyclic cocycle.

Proof. Proposition 3.2 combined with the Hölder inequality ensures that the r.h.s. of (22) is finite on \mathcal{A}_{loc} . The expressions defined in (21) exist almost surely and belongs to the kernel of the Hochschild coboundary map. This is an algebraic property automatically extends to τ_d . \square

Definition 4.2. The cohomology class in the cyclic cohomology of \mathcal{A}_{loc} of the cycle cocycle τ_d defined in (22) is called the Chern character of the family of Fredholm modules $(\mathcal{H}, F_{\mathbf{x}_0}, \pi_\omega)$ and is denoted by Ch_* .

Recall that $\text{GL}_\infty(\mathcal{A}_{\text{loc}})$ is the inductive limit (with natural embeddings) of the groups $\text{GL}_q(\mathcal{A}_{\text{loc}})$ of invertible matrices of size $q \times q$ with entries from \mathcal{A}_{loc} . Then $\text{GL}_\infty(\mathcal{A}_{\text{loc}})_0$ is the connected component of the unity in $\text{GL}_\infty(\mathcal{A}_{\text{loc}})$ and the topological version of $K_1(\mathcal{A}_{\text{loc}})$ is defined as $\text{GL}_\infty(\mathcal{A}_{\text{loc}})/\text{GL}_\infty(\mathcal{A}_{\text{loc}})_0$. As τ_d is continuous

on \mathcal{A}_{loc} (which is endowed with the norm $\|\cdot\|_2$), it naturally pairs with $K_1(\mathcal{A}_{\text{loc}})$. The following proposition transposes the fundamental properties of cyclic cocycles to the present context.

Proposition 4.3. (i) *The map*

$$v \in \text{GL}_\infty(\mathcal{A}_{\text{loc}}) \mapsto (\tau_d \# \text{Tr})(v^{-1} - 1, v - 1, \dots, v^{-1} - 1, v - 1), \quad (23)$$

is constant on the equivalence class $[v]$ of v in $K_1(\mathcal{A}_{\text{loc}})$.

(ii) *This constant value remains unchanged if τ_d is replaced by any other representative from its cohomology class. Hence (23) defines natural pairing $\langle [v], \text{Ch}_* \rangle$ between $K_1(\mathcal{A}_{\text{loc}})$ and the Chern character.*

(iii) *The pairing is integral and given by an almost surely constant index of a co-variant family of Fredholm operators. Specifically, if $v \in \text{GL}_q(\mathcal{A}_{\text{loc}})$, then*

$$\langle [v], \text{Ch}_* \rangle = \text{Ind}(E_{\mathbf{x}_0}^q \pi_\omega^q(v) E_{\mathbf{x}_0}^q) \in \mathbb{Z}, \quad (24)$$

where $F_{\mathbf{x}_0}^q = F_{\mathbf{x}_0} \otimes \mathbf{1}_q$ and $E_{\mathbf{x}_0}^q = \frac{1}{2}(\mathbf{1} + F_{\mathbf{x}_0}^q)$, and $\pi_\omega^q = \pi_\omega \otimes \text{id}$ is the representation of \mathcal{A}_{loc} on $\mathcal{H}^q = \mathcal{H} \otimes \mathbb{C}^q$. The index is almost surely independent of ω and \mathbf{x}_0 .

Proof. Claims (i) and (ii) are listed on p. 230 of [5], while (iii) will require a slight modification of the arguments on p. 303. First of all, by the same reasoning $(d+1)$ -summability ensures that $E_{\mathbf{x}_0}^q \pi_\omega^q(v) E_{\mathbf{x}_0}^q$ is a Fredholm operator almost surely. Then by the Calderon-Fedosov formula

$$\text{Ind}(E_{\mathbf{x}_0}^q \pi_\omega^q(v) E_{\mathbf{x}_0}^q) = \lambda_d \text{Tr}'(\pi_\omega^q(v^{-1} - 1)[F_{\mathbf{x}_0}^q, \pi_\omega^q(v - 1)] \cdots [F_{\mathbf{x}_0}^q, \pi_\omega^q(v - 1)]).$$

The r.h.s. enters in the definition of τ_d . Its average over ω and \mathbf{x}_0 remains integer because it will be shown next that the indices are almost surely invariant with respect to ω and \mathbf{x}_0 . Let us first examine the behavior of the index with ω . Since \mathbf{P} is ergodic, it is sufficient to check constancy of the index along every orbit, namely compare the indices of $E_{\mathbf{x}_0} \pi_\omega(v) E_{\mathbf{x}_0}$ and $E_{\mathbf{x}_0} \pi_{T_{\mathbf{a}} \omega}(v) E_{\mathbf{x}_0}$ for arbitrary $\mathbf{a} \in \mathbb{Z}^d$. Since the index is invariant to conjugations with unitaries, one only needs to check equality of the indices of $E_{\mathbf{x}_0} \pi_\omega(v) E_{\mathbf{x}_0}$ and $E_{\mathbf{a} + \mathbf{x}_0} \pi_\omega(v) E_{\mathbf{a} + \mathbf{x}_0}$. But

$$\begin{aligned} & E_{\mathbf{a} + \mathbf{x}_0} \pi_\omega(v) E_{\mathbf{a} + \mathbf{x}_0} - E_{\mathbf{x}_0} \pi_\omega(v) E_{\mathbf{x}_0} \\ &= \frac{1}{2}[(F_{\mathbf{a} + \mathbf{x}_0} - F_{\mathbf{x}_0}) \otimes \mathbf{1}] \pi_\omega(v) E_{\mathbf{a} + \mathbf{x}_0} + \frac{1}{2} E_{\mathbf{x}_0} \pi_\omega(v) [(F_{\mathbf{a} + \mathbf{x}_0} - F_{\mathbf{x}_0}) \otimes \mathbf{1}]. \end{aligned}$$

The operator

$$F_{\mathbf{a} + \mathbf{x}_0} - F_{\mathbf{x}_0} = \widehat{(\mathbf{a} + \mathbf{x}_0 + \mathbf{X} - \mathbf{x}_0 + \mathbf{X})} \cdot \boldsymbol{\sigma}$$

has the singular values

$$|\widehat{(\mathbf{a} + \mathbf{x}_0 + \mathbf{x} - \mathbf{x}_0 + \mathbf{x})}|,$$

which behave as $|\mathbf{a} + (\mathbf{a} \cdot \hat{\mathbf{x}})\hat{\mathbf{x}}|/|\mathbf{x}|$ in the limit $|\mathbf{x}| \rightarrow \infty$, hence this operator is compact. The compact stability of the index now allows to conclude. The invariance in \mathbf{x}_0 follows by a similar argument. \square

The main result of this work is the following local formula for the Chern character.

Theorem 4.4. *For $f_0, \dots, f_d \in \mathcal{A}_{\text{loc}}$,*

$$\tau_d(f_0, \dots, f_d) = \Lambda_d \sum_{\rho} (-1)^{\rho} \mathcal{T} \left(f_0 \prod_{i=1}^d \partial_{\rho_i} f_i \right), \quad \Lambda_d = \frac{i(-i\pi)^{\frac{d-1}{2}}}{d!!}. \quad (25)$$

Proof. Due to the continuity of both side w.r.t. $\|\cdot\|_2$, it sufficient to check the identity for $f_0, \dots, f_d \in \mathcal{A}_0$. Let us begin by writing out the graded trace and graded commutator in the defining formula (22) of τ_d . Using $F_{\mathbf{x}}[F_{\mathbf{x}}, \pi_{\omega}(f)] = -[F_{\mathbf{x}}, \pi_{\omega}(f)]F_{\mathbf{x}}$ one obtains

$$\tau_d(f_0, \dots, f_d) = \frac{\lambda_d}{2} \int_{\mathcal{C}^d} d\mathbf{x}_0 \int_{\Omega} \mathbf{P}(d\omega) \operatorname{Tr} \left(F_{\mathbf{x}_0} \prod_{i=0}^d [F_{\mathbf{x}_0}, \pi_{\omega}(f_i)] \right).$$

Now the trace is written out using the covariance property and the trace $\operatorname{tr}_{\sigma, N}$ over the fiber $\mathbb{C}^N \otimes \operatorname{Cliff}(d)$ at the origin $\mathbf{0}$:

$$\begin{aligned} \tau_d(f_0, \dots, f_d) &= \frac{\lambda_d}{2} \int_{\mathcal{C}^d} d\mathbf{x}_0 \int_{\Omega} \mathbf{P}(d\omega) \sum_{\mathbf{x} \in \mathbb{Z}^d} \operatorname{tr}_{\sigma, N} \left(F_{\mathbf{x}_0 + \mathbf{x}} \prod_{i=0}^d [F_{\mathbf{x}_0 + \mathbf{x}}, \pi_{T_{\mathbf{x}}\omega}(f_i)] \right) \\ &= \frac{\lambda_d}{2} \int_{\mathbb{R}^d} d\mathbf{x} \int_{\Omega} \mathbf{P}(d\omega) \operatorname{tr}_{\sigma, N} \left(F_{\mathbf{x}} \prod_{i=0}^d [F_{\mathbf{x}}, \pi_{\omega}(f_i)] \right), \end{aligned}$$

where in the second equality the invariance of \mathbf{P} was used. Writing out the commutator $[F_{\mathbf{x}}, \pi_{\omega}(f_i)]$ and using again $F_{\mathbf{x}}[F_{\mathbf{x}}, \pi_{\omega}(f)] = -[F_{\mathbf{x}}, \pi_{\omega}(f)]F_{\mathbf{x}}$ as well as $F_{\mathbf{x}}^2 = \mathbf{1}$ then leads to

$$\tau_d(f_0, \dots, f_d) = \lambda_d \int_{\mathbb{R}^d} d\mathbf{x} \int_{\Omega} \mathbf{P}(d\omega) \operatorname{tr}_{\sigma, N} \left(\pi_{\omega}(f_0) \prod_{i=1}^d [F_{\mathbf{x}}, \pi_{\omega}(f_i)] \right).$$

Next let us insert partitions of unity using the projections $\chi_{\mathbf{x}}$ defined in (18):

$$\tau_d(f_0, \dots, f_d) = \lambda_d \int_{\mathbb{R}^d} d\mathbf{x} \int_{\Omega} \mathbf{P}(d\omega) \sum_{\mathbf{x}_i \in \mathbb{Z}^d} \operatorname{tr}_{\sigma, N} \left(\pi_{\omega}(f_0) \prod_{i=1}^d \chi_{\mathbf{x}_i} [F_{\mathbf{x}}, \pi_{\omega}(f_i)] \chi_{\mathbf{x}_{i+1}} \right),$$

where \mathbf{x}_{2n+2} is fixed at the origin. The sum over \mathbf{x}_i contains a finite number of terms, so it can be interchanged with the integrals to continue to

$$\begin{aligned} \tau_d(f_0, \dots, f_d) &= \lambda_d \sum_{\mathbf{x}_i \in \mathbb{Z}^d} \int_{\mathbb{R}^d} d\mathbf{x} \operatorname{tr}_{\sigma} \left(\prod_{i=1}^d (\widehat{\mathbf{x}_i + \mathbf{x} - \mathbf{x}_{i+1} + \mathbf{x}}) \cdot \boldsymbol{\sigma} \right) \\ &\quad \times \int_{\Omega} \mathbf{P}(d\omega) \operatorname{tr}_N \left(\pi_{\omega}(f_0) \prod_{i=1}^d \chi_{\mathbf{x}_i}(\pi_{\omega}(f_i)) \chi_{\mathbf{x}_{i+1}} \right). \end{aligned}$$

Here $\operatorname{tr}_{\sigma}$ and tr_N are as in the proof of Proposition 3.2, and actually the identity is analogous to (19) in that proof. Now the integral in the first line can be evaluated with the identity from Lemma 4.5. Expressing the sum over \mathbf{x}_i again with the position operator X_i and connecting the constants λ_d and Λ_d then leads to

$$\tau_d(f_0, \dots, f_d) = i^d \Lambda_d \int_{\Omega} \mathbf{P}(d\omega) \sum_{\rho \in S_d} (-1)^{\rho} \operatorname{tr}_N \left(\pi_{\omega}(f_0) \prod_{i=1}^d X_{\rho_i} \pi_{\omega}(f_i) \right).$$

Due to the anti-symmetrizing factor $(-1)^{\rho}$, one can actually form commutators:

$$\tau_d(f_0, \dots, f_d) = i^d \Lambda_d \int_{\Omega} \mathbf{P}(d\omega) \sum_{\rho \in S_d} (-1)^{\rho} \operatorname{tr}_N \left(\pi_{\omega}(f_0) \prod_{i=1}^d [X_{\rho_i}, \pi_{\omega}(f_i)] \right),$$

where it was also used that only one summand of the last commutator contributes. This expression can finally be rewritten with the non-commutative analysis tools to complete the proof of (25). \square

Lemma 4.5 (Key geometric identity). *Let $\mathbf{x}_1, \dots, \mathbf{x}_{d+1} \in \mathbb{R}^d$ with $\mathbf{x}_{d+1} = \mathbf{0}$ fixed at the origin. Then:*

$$\int_{\mathbb{R}^d} d\mathbf{x} \operatorname{tr}_\sigma \left(\prod_{i=1}^d (\widehat{\mathbf{x}_i + \mathbf{x} - \mathbf{x}_{i+1} + \mathbf{x}}) \cdot \boldsymbol{\sigma} \right) = \frac{2^d (-i\pi)^{\frac{d-1}{2}}}{d!!} \sum_{\rho \in S_d} (-1)^\rho \prod_{i=1}^d x_{i, \rho_i}, \quad (26)$$

where $x_{i,j}$ denotes the j th component of $\mathbf{x}_i = (x_{i,1}, \dots, x_{i,d})$.

Remark. A similar identity for even dimensions and even Clifford algebras has been found in [12]. \diamond

Proof. Let us first list several useful identities for the σ matrices:

- (i) $\sigma_1 \sigma_2 \cdots \sigma_d = \pm (-i)^{\frac{d-1}{2}} \mathbf{1}$ (we choose the representation such that “+” occurs);
- (ii) $\operatorname{tr}_\sigma (\sigma_{\rho_1} \cdots \sigma_{\rho_q}) = 0$ if q odd and $q < d$;
- (iii) $\operatorname{tr}_\sigma (\sigma_{\rho_1} \cdots \sigma_{\rho_d}) = 0$ unless ρ is a permutation of $1, 2, \dots, d$;
- (iv) $\operatorname{tr}_\sigma (\sigma_{\rho_1} \cdots \sigma_{\rho_d}) = (-2i)^{\frac{d-1}{2}} (-1)^\rho$ if ρ is such a permutation.

All these identities follow from the defining relations of the Clifford algebra. Notice that the product in the first identity commutes with all σ ’s hence it must be proportional to the identity matrix. Also, the square of the product equals $(-1)^{\frac{d-1}{2}} \mathbf{1}$, which enables one to establish the constant in the first identity. The second identity follows from the fact that a conjugation with a σ which is absent in that product of σ ’s (here is where the condition $q < d$ enters) changes the sign of the trace but in the same time the trace is invariant to conjugations. If ρ is not a permutation in the third identity, then pairs of identical σ ’s can be erased from the product and the identity then follows from the second identity. The fourth identity is evident.

Using these identities, one can next establish

$$\operatorname{tr}_\sigma \left(\prod_{i=1}^d (\mathbf{y}_i \cdot \boldsymbol{\sigma}) \right) = (-2i)^{\frac{d-1}{2}} d! \operatorname{Vol}[\mathbf{0}, \mathbf{y}_1, \dots, \mathbf{y}_d], \quad (27)$$

for any set of points $\mathbf{y}_1, \dots, \mathbf{y}_d \in \mathbb{R}^d$ and with $[\mathbf{y}_0, \mathbf{y}_1, \dots, \mathbf{y}_d]$ denoting the corresponding simplex, and $\operatorname{Vol}[\dots]$ the oriented volume of the simplex. Indeed, expanding and taking into account (iii), one has

$$\operatorname{tr}_\sigma \left(\prod_{i=1}^d (\mathbf{y}_i \cdot \boldsymbol{\sigma}) \right) = \sum_{\rho \in S_d} y_{1, \rho_1} \cdots y_{d, \rho_d} \operatorname{tr}_\sigma (\sigma_{\rho_1} \cdots \sigma_{\rho_d}),$$

and from (iv)

$$\operatorname{tr}_\sigma \left(\prod_{i=1}^d (\mathbf{y}_i \cdot \boldsymbol{\sigma}) \right) = (-2i)^{\frac{d-1}{2}} \operatorname{Det}[\mathbf{y}_1, \dots, \mathbf{y}_d],$$

where inside the determinant is the $d \times d$ -matrix of columns $\mathbf{y}_1, \dots, \mathbf{y}_d$. The volume of a simplex can be computed with the formula:

$$\operatorname{Vol}[\mathbf{y}_0, \mathbf{y}_1, \dots, \mathbf{y}_d] = \frac{1}{d!} \operatorname{Det} \begin{bmatrix} \mathbf{y}_0 & \mathbf{y}_1 & \cdots & \mathbf{y}_d \\ 1 & 1 & \cdots & 1 \end{bmatrix},$$

hence Eq. 27 follows by setting $\mathbf{y}_0 = \mathbf{0}$ above.

For the computation of the l.h.s. of (26) let us set

$$I(\mathbf{x}) = \text{tr}_\sigma \left(\prod_{i=1}^d (\widehat{\mathbf{x}_i - \mathbf{x} - \mathbf{x}_{i+1} - \mathbf{x}}) \cdot \boldsymbol{\sigma} \right) .$$

Writing all terms one first finds

$$I(\mathbf{x}) = \sum_{j=1}^{d+1} (-1)^j \text{tr}_\sigma \left((\widehat{\mathbf{x}_1 - \mathbf{x} \cdot \boldsymbol{\sigma}}) \cdots \underline{(\widehat{\mathbf{x}_j - \mathbf{x} \cdot \boldsymbol{\sigma}})} \cdots (\widehat{\mathbf{x}_{d+1} - \mathbf{x} \cdot \boldsymbol{\sigma}}) \right) ,$$

where the underline designates a factor which is omitted. Thus with (27)

$$I(\mathbf{x}) = (-2i)^{\frac{d-1}{2}} d! \sum_{j=1}^{d+1} (-1)^j \text{Vol}[\mathbf{0}, \widehat{\mathbf{x}_1 - \mathbf{x}}, \dots, \underline{\widehat{\mathbf{x}_j - \mathbf{x}}}, \dots, \widehat{\mathbf{x}_{d+1} - \mathbf{x}}] .$$

The vertices can be re-ordered and it is convenient to translate the whole simplex, to continue:

$$I(\mathbf{x}) = (-2i)^{\frac{d-1}{2}} d! \sum_{j=1}^{d+1} \text{Vol}[\mathbf{x} + \widehat{\mathbf{x}_1 - \mathbf{x}}, \dots, \mathbf{x}, \dots, \mathbf{x} + \widehat{\mathbf{x}_{d+1} - \mathbf{x}}] ,$$

where the vertex \mathbf{x} is located at the j -th position. At this point, it is useful to introduce the notations

$$\mathfrak{S}_j(\mathbf{x}) = [\mathbf{x} + \widehat{\mathbf{x}_1 - \mathbf{x}}, \dots, \mathbf{x}, \dots, \mathbf{x} + \widehat{\mathbf{x}_{d+1} - \mathbf{x}}]$$

and

$$\mathfrak{S} = [\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{d+1}] ,$$

where $\mathbf{x}_{d+1} = \mathbf{0}$. The orientations of these simplexes are the same because each $\mathfrak{S}_j(\mathbf{x})$ can be continuously deformed into \mathfrak{S} without reducing the volume to zero. Now note that, for arbitrarily selected j , all vertices

$$\mathbf{x} + \widehat{\mathbf{x}_1 - \mathbf{x}}, \dots, \underline{\widehat{\mathbf{x}_j - \mathbf{x}}}, \dots, \mathbf{x} + \widehat{\mathbf{x}_{d+1} - \mathbf{x}}$$

of the simplex $\mathfrak{S}_j(\mathbf{x})$ are located on the unit sphere centered at \mathbf{x} . As such, the facets of $\mathfrak{S}_j(\mathbf{x})$ stemming from \mathbf{x} define a d -dimensional sector of the unit ball. This sector will be denoted by $\mathfrak{B}_j(\mathbf{x})$ and its orientation is taken to be the same as of $\mathfrak{S}_j(\mathbf{x})$. The entire unit ball will be denoted by \mathfrak{B} and its orientation will be taken to be the same as that of \mathfrak{S} . One key fact is that:

$$\text{Vol}(\mathfrak{S}_j(\mathbf{x})) - \text{Vol}(\mathfrak{B}_j(\mathbf{x})) \sim |\mathbf{x}|^{-(d+1)} \quad \text{as } |\mathbf{x}| \rightarrow \infty .$$

This enables one to break the integral into two terms:

$$\begin{aligned} \int_{\mathbb{R}^d} d\mathbf{x} I(\mathbf{x}) &= (-2i)^{\frac{d-1}{2}} d! \sum_{j=1}^{d+1} \int_{\mathbb{R}^d} d\mathbf{x} [\text{Vol}(\mathfrak{S}_j(\mathbf{x})) - \text{Vol}(\mathfrak{B}_j(\mathbf{x}))] \\ &\quad + (-2i)^{\frac{d-1}{2}} d! \int_{\mathbb{R}^d} d\mathbf{x} \sum_{j=1}^{d+1} \text{Vol}(\mathfrak{B}_j(\mathbf{x})) . \end{aligned}$$

At this point let us note that

$$\int_{\mathbb{R}^d} d\mathbf{x} [\text{Vol}(\mathfrak{S}_j(\mathbf{x})) - \text{Vol}(\mathfrak{B}_j(\mathbf{x}))] = 0 ,$$

which is a consequence of the odd-symmetry of the integrand relative to the inversion of \mathbf{x} relative to the center of the facet $\mathbf{x}_1, \dots, \mathbf{x}_j, \dots, \mathbf{x}_{d+1}$ of \mathfrak{S} . Furthermore:

$$\sum_{j=1}^{d+1} \text{Vol}(\mathfrak{B}_j(\mathbf{x})) = \begin{cases} \text{Vol}(\mathfrak{B}) & \text{if } \mathbf{x} \text{ inside } \mathfrak{S}, \\ 0 & \text{if } \mathbf{x} \text{ outside } \mathfrak{S}, \end{cases}$$

which follows because the solid angles corresponding to the facets of the simplex \mathfrak{S} , as viewed from \mathbf{x} , add up to the full solid angle if \mathbf{x} is inside the simplex, and they add up to zero if \mathbf{x} is outside the simplex. Hence

$$\int_{\mathbb{R}^d} d\mathbf{x} I(\mathbf{x}) = (-2i)^{\frac{d-1}{2}} d! \text{Vol}(\mathfrak{B}) |\text{Vol}(\mathfrak{S})|.$$

Now the orientations of \mathfrak{B} and \mathfrak{S} are the same so that

$$\text{Vol}(\mathfrak{B}) |\text{Vol}(\mathfrak{S})| = |\text{Vol}(\mathfrak{B})| \text{Vol}(\mathfrak{S}) = \frac{2(2\pi)^{\frac{d-1}{2}}}{d!!} \frac{1}{d!} \det(\mathbf{x}_1, \dots, \mathbf{x}_d),$$

and the identity follows. \square

The numerical invariant generated by the pairing of the Chern character with the K_1 group can be rightfully called the non-commutative odd Chern number associated to v :

$$\text{Ch}_d(v) = \frac{i(2\pi)^{\frac{d-1}{2}}}{d!!} \sum_{\rho} (-1)^{\rho} \mathcal{T} \left(\prod_{i=1}^d v^{-1} \partial_{\rho_i} v \right). \quad (28)$$

In the operator representation, Ch_d takes exactly the form presented in (3). Furthermore, when Ω reduces to just one point (*i.e.* for translationally invariant systems), (3) is nothing but the real-space representation of the classical odd Chern number of (3) over the torus. Below we list the properties of the non-commutative Chern number which follow directly from the theory developed above.

Theorem 4.6 (Quantization and Invariance of Ch_d). *Consider the algebra of covariant observables for homogeneous lattice models over $\ell^2(\mathbb{Z}^d, \mathbb{C}^N)$, and the subalgebra \mathcal{A}_{loc} of the localized observables defined in Proposition 3.1. Then:*

(i) *For any invertible v from \mathcal{A}_{loc} :*

$$\text{Ch}_d(v) = \text{Ind}(E_{\mathbf{x}_0} \pi_{\omega}(v) E_{\mathbf{x}_0}) \in \mathbb{Z},$$

where the index is with probability 1 independent of \mathbf{x}_0 and ω .

(ii) *$\text{Ch}_d(v_t)$ remains pinned at a quantized value for any homotopy v_t of invertibles inside \mathcal{A}_{loc} endowed with the norm $\|\cdot\|_2$.*

5. APPLICATION TO STABILITY OF THE TOPOLOGICAL PHASES

This section provides the details supporting the claims (i), (ii) and (iii) made in the introduction. For this purpose, let $(H_{\omega})_{\omega \in \Omega}$ be a covariant family of tight-binding Hamiltonian on $\ell^2(\mathbb{Z}^d, \mathbb{C}^{2N})$ as described in Section 2.2. In particular, the covariance relation (11) holds as well as the chiral symmetry (4) so that by (13) there are associated unitaries $(U_{\omega})_{\omega \in \Omega}$ which are also covariant. In order to apply Theorem 1.1, it remains to show that this family of unitaries satisfies the bound (2), or equivalently in the terminology of Section 3, that it defines an element in the subalgebra \mathcal{A}_{loc} . This will be assured by the following

Aizenman-Molchanov localization regime for an open interval $\Delta \subset \mathbb{R}$: For some $s \in (0, 1)$ there exist positive constants C_s and β_s such that for all $E \in \Delta$

$$\int_{\Omega} \mathbf{P}(d\omega) \|\langle \mathbf{x} | (H_{\omega} - E \pm i0^+)^{-1} | \mathbf{y} \rangle\|^s \leq C_s e^{-\beta_s |\mathbf{x} - \mathbf{y}|}. \quad (29)$$

That this bound actually holds at weak to moderate disorder for an interval Δ lying in a gap of the unperturbed Hamiltonian is proved in [2] for the scalar model and for matrix-valued models considered in this work in [6]. Furthermore, both works show explicitly that there is a regime of moderate disorder for which Δ is entirely part of the spectrum. While [6] is formulated for BdG models, the same argument implies that for chiral models there is a regime where the Fermi energy $E_F = 0$ lies in an interval of the Aizenman-Molchanov localization regime. How to deduce bounds on the matrix elements of the Fermi projections corresponding to Fermi energies in Δ has been shown in various papers [1, 2] and this immediately implies the bound on $(U_{\omega})_{\omega \in \Omega}$. As in [12] we follow the basic idea of [1] to represent the Fermi projection as regularized contour integral. Once this is done, Theorem 1.1 applies and combined with a standard homotopy argument the following can be deduced:

Theorem 5.1. *Let $(H_{\omega})_{\omega \in \Omega}$ be a covariant family of chiral unitary, tight-binding Hamiltonian on $\ell^2(\mathbb{Z}^d, \mathbb{C}^{2N})$. Suppose that the Fermi level $E_F = 0$ lies in an open interval Δ for which the Aizenman-Molchanov localization regime holds. Then the covariant family of unitaries $U = (U_{\omega})_{\omega \in \Omega}$ associated by (13) lies in \mathcal{A}_{loc} so that its non-commutative odd Chern number is well defined and given by an almost surely constant index $\text{Ch}_d(U) = \text{Ind}(E_{\mathbf{x}_0} U_{\omega} E_{\mathbf{x}_0})$.*

Furthermore, let $t \in [0, 1] \mapsto H_{\omega}(t) = H_{\omega} + t \delta H_{\omega}$ be a deformation of the Hamiltonian such that each $H_{\omega}(t)$ is chiral and has the Fermi level in a region of localized states, while the perturbation δH_{ω} has finite range and satisfies the bound $\sup_{\omega, \mathbf{x}} \|\langle \mathbf{0} | \delta H_{\omega} | \mathbf{x} \rangle\| < \infty$. Then for $U(t)$ defined as above, the odd Chern number $\text{Ch}_d(U(t))$ is constant in t .

Proof. First of all, due to (13) it is sufficient to show that $\mathbf{1} - 2P_{\omega}$ satisfies the bound (2) or equivalently (16). As already hinted at above, this will deduced using as in [1] the identity

$$\langle \mathbf{0} | \mathbf{1} - 2P_{\omega} | \mathbf{x} \rangle = \delta_{\mathbf{0}, \mathbf{x}} \mathbf{1}_{2N} - \oint_{\gamma} \frac{dz}{i\pi} \langle \mathbf{0} | (z - H_{\omega})^{-1} | \mathbf{x} \rangle, \quad (30)$$

where the contour γ encircles the negative spectrum and crosses the real axis at $E_F = 0$. Let us add a few comments to justify this formula. First of all, it is well-known that $E_F = 0$ is \mathbf{P} -almost surely not an eigenvalue and that the weak limits $\langle \mathbf{0} | (E \pm i0 - H_{\omega})^{-1} | \mathbf{x} \rangle$ exist for Lebesgue almost all E . Thus almost surely, the integral on the r.h.s. exists and actually defines as a weak integral an operator. The standard argument showing that Riesz projections are indeed orthogonal projections can be repeated and, furthermore, by using the spectral representation of H_{ω} one deduces that this projection is indeed the Fermi projection.

To bound the r.h.s. of (30), let us first recall that a standard Combes-Thomas estimate (not pending on localization estimates) implies that an exponential bound on $\langle \mathbf{0} | (z - H_{\omega})^{-1} | \mathbf{x} \rangle$ holds, albeit with decay rate proportional to $\Im m(z)$. Combined with (29) this implies that subexponential decay holds uniformly along the path γ .

Thus, for $\mathbf{x} \neq \mathbf{0}$,

$$\begin{aligned} \int_{\Omega} \mathbf{P}(d\omega) \|\langle \mathbf{0} | \mathbf{1} - 2P_{\omega} | \mathbf{x} \rangle\| &\leq \int_{\gamma} \frac{dz}{\pi} |\Im m(z)|^{s-1} \int_{\Omega} \mathbf{P}(d\omega) \|\langle \mathbf{0} | (z - H_{\omega})^{-1} | \mathbf{x} \rangle\|^s \\ &\leq C e^{-\beta_s |\mathbf{x}|}. \end{aligned}$$

As to the second claim, according to Theorem 4.6 it is sufficient to show that $U(t)$ depends continuously on t w.r.t. the GNS norm. Again this follows from the continuity of $P(t) = (P_{\omega}(t))_{\omega \in \Omega}$. But (30) combined with the resolvent identity shows

$$\begin{aligned} \|\langle \mathbf{0} | P_{\omega}(t) - P_{\omega}(t') | \mathbf{x} \rangle\| &\leq \int_{\gamma} \frac{dz}{2\pi} \|\langle \mathbf{0} | (z - H_{\omega}(t))^{-1} (t - t') \delta H_{\omega}(z - H_{\omega}(t'))^{-1} | \mathbf{x} \rangle\| \\ &\leq C |t - t'| \int_{\gamma} \frac{dz}{|\Im m(z)|^{1-s}} \sum_{\mathbf{x}', \mathbf{x}''} \|\langle \mathbf{0} | (z - H_{\omega})^{-1} | \mathbf{x}' \rangle\|^{\frac{s}{2}} \|\langle \mathbf{x}'' | (z - H_{\omega})^{-1} | \mathbf{x} \rangle\|^{\frac{s}{2}}, \end{aligned}$$

where in the second inequality the uniform bound on the matrix elements of δH_{ω} was used, and the sum runs over all $\mathbf{x}', \mathbf{x}'' \in \mathbb{Z}^d$ such that $|\mathbf{x}' - \mathbf{x}''| \leq R$ where R is the range of δH_{ω} . Now by the Cauchy-Schwarz inequality and the bound (29) holding all along γ

$$\begin{aligned} &\int_{\Omega} \mathbf{P}(d\omega) \|\langle \mathbf{0} | P_{\omega}(t) - P_{\omega}(t') | \mathbf{x} \rangle\| \\ &\leq C' |t - t'| \int_{\gamma} \frac{dz}{|\Im m(z)|^{1-s}} \sum_{\mathbf{x}', \mathbf{x}''} e^{-\frac{1}{2}\beta_s |\mathbf{x}'|} e^{-\frac{1}{2}\beta_s |\mathbf{x}'' - \mathbf{x}|} \\ &\leq C'' |t - t'| e^{-\frac{1}{4}\beta_s |\mathbf{x}|}. \end{aligned}$$

Thus implies

$$\|P(t) - P(t')\|_2^2 \leq \sum_{\mathbf{x}'} \int_{\Omega} \mathbf{P}(d\omega) \|\langle \mathbf{0} | P_{\omega}(t) - P_{\omega}(t') | \mathbf{x} \rangle\| \leq C''' |t - t'|,$$

and therefore the required continuity. \square

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PHYSICS DEPARTMENT, YESHIVA UNIVERSITY, NEW YORK, NY, 10016, USA.

E-mail address: `prodan@yu.edu`

DEPARTMENT MATHEMATIK, UNIVERSITÄT ERLANGEN-NÜRNBERG, 91058 ERLANGEN, GERMANY

E-mail address: `schuba@mi.uni-erlangen.de`